Phase operators, temporally stable phase states, mutually unbiased bases and exactly solvable quantum systems

Phase operators, temporally stable phase states, mutually unbiased bases and exactly solvable quantum systems

M Daoud[1,2,3,4] **and M R Kibler**[1,2,3]

[1] Université Lyon, F-69622, Lyon, France
[2] Université Claude Bernard Lyon-1, Villeurbanne, France
[3] CNRS/IN2P3, Institut de Physique Nucléaire de Lyon, France
[4] Faculté des Sciences, Agadir, Morocco

E-mail: m_daoud@hotmail.com and m.kibler@ipnl.in2p3.fr

Abstract

We introduce a one-parameter-generalized oscillator algebra $\mathcal{A}_\kappa$ (that covers the case of the harmonic oscillator algebra) and discuss its finite- and infinite-dimensional representations according to the sign of the parameter κ. We define an (Hamiltonian) operator associated with $\mathcal{A}_\kappa$ and examine the degeneracies of its spectrum. For the finite (when $\kappa < 0$) and the infinite (when $\kappa \geqslant 0$) representations of $\mathcal{A}_\kappa$, we construct the associated phase operators and build temporally stable phase states as eigenstates of the phase operators. To overcome the difficulties related to the phase operator in the infinite-dimensional case and to avoid the degeneracy problem for the finite-dimensional case, we introduce a truncation procedure which generalizes the one used by Pegg and Barnett for the harmonic oscillator. This yields a truncated-generalized oscillator algebra $\mathcal{A}_{\kappa,s}$, where s denotes the truncation order. We construct two types of temporally stable states for $\mathcal{A}_{\kappa,s}$ (as eigenstates of a phase operator and as eigenstates of a polynomial in the generators of $\mathcal{A}_{\kappa,s}$). Two applications are considered in this paper. The first concerns physical realizations of $\mathcal{A}_\kappa$ and $\mathcal{A}_{\kappa,s}$ in the context of one-dimensional quantum systems with finite (Morse system) or infinite (Pöschl–Teller system) discrete spectra. The second deals with mutually unbiased bases used in quantum information.

PACS numbers: 03.65.Fd, 03.65.Ta, 03.65.Ud, 02.20.Qs

1. Introduction

It is well known that the usual model for the quantized single modes of the electromagnetic field is the harmonic oscillator with an infinity of states. The infinite-dimensional character

of the representation space of the corresponding oscillator algebra constitutes a drawback to define a phase operator in a consistent way [1–3]. In order to get rid of this difficulty, Pegg and Barnett suggested to truncate to some finite (but arbitrarily large) order the infinite-dimensional representation space of the oscillator algebra [4]. Their approach also provided a valid way for calculating the so-called phase states (the eigenvectors of the phase operator). In the same vein, Vourdas proposed a definition of a phase operator for $su(2)$ and calculated its eigenstates without a truncation procedure since $su(2)$ admits finite-dimensional unitary irreducible representations [5]. He also constructed a phase operator and its eigenstates for $su(1, 1)$, without a truncation procedure although $su(1, 1)$ admits infinite-dimensional unitary irreducible representations [5].

The main aim of the present work is to develop a method to build unitary phase operators[5] and temporally stable phase states for some exactly solvable quantum systems. Various algebraic structures were used to construct (temporary stable or not) coherent states in connection with some quantum systems [6–10]. The construction of temporally stable phase states to be developed in this work is based on a generalized oscillator algebra which takes its root in [11, 12]. This algebra was introduced to construct isospectral shape invariant potentials in the framework of fractional supersymmetry.

A second facet of this work is to show that the obtained temporally stable phase states can be used to generate mutually unbiased bases (MUBs). Such bases are of considerable interest in quantum information and were recently investigated from an angular momentum approach [13, 14]. It is not the purpose of this paper to deal with unsolved problems concerning MUBs but to give a way to construct MUBs from temporally stable states associated with some exactly solvable systems.

The paper is organized as follows. Section 2 is devoted to the generalized oscillator algebra $\mathcal{A}_\kappa$. Temporally stable phase states associated with $\mathcal{A}_\kappa$ are studied in section 3. Section 4 deals with the truncated oscillator algebra $\mathcal{A}_{\kappa,s}$ and the corresponding phase states. As a first application, the derivation of MUBs from phase states is developed in section 5. A second application is made in section 6 to some exactly solvable quantum systems.

The notations are standard. Let us simply mention that: $\delta_{a,b}$ stands for the Kronecker symbol of a and b, I for the identity operator, $A^\dagger$ for the adjoint of the operator A, and $[A, B]$ and $\{A, B\}$, respectively, for the commutator and the anticommutator of the operators A and B. We use a notation of type $|\psi\rangle$ for a vector in a Hilbert space and denote by $\langle\phi|\psi\rangle$ and $|\phi\rangle\langle\psi|$, respectively, the inner and outer products of the vectors $|\psi\rangle$ and $|\phi\rangle$.

2. Generalized oscillator algebra

2.1. The algebra $\mathcal{A}_\kappa$

Let $\mathcal{A}_\kappa$ be the algebra spanned by the three linear operators a^-, a^+ and N, satisfying the following relations:

$$[a^-, a^+] = I + 2\kappa N \qquad [N, a^\pm] = \pm a^\pm \qquad (a^-)^\dagger = a^+ \qquad N^\dagger = N, \qquad (1)$$

where κ is a real parameter. Note that, for $\kappa = 0$, the algebra $\mathcal{A}_0$ is nothing but the usual harmonic oscillator algebra. The operators a^-, a^+ and N in (1) generalize the annihilation, creation and number operators used for the harmonic oscillator. Therefore, the algebra $\mathcal{A}_\kappa$ shall be called the generalized oscillator algebra. This algebra turns out to be a particular case of the generalized Weyl–Heisenberg algebra W_k introduced in [11, 12] and not to be confused

---

[5] We deal here with *unitary* rather than *Hermitian* phase operators. The two kinds of operators are related via an exponentiation trick.

with the Lie algebra of the Heisenberg–Weyl group $HW(\mathbb{R})$ used in quantum information [14]. In fact, $\mathcal{A}_\kappa$ is identical to W_k with

$$k = 1 \qquad f_0(N) = aN + b \qquad \frac{1}{\sqrt{b}} X_\pm = a^\pm \qquad \kappa = \frac{1}{2}\frac{a}{b}, \tag{2}$$

where the operators $f_0(N)$ and $X_\pm$, and the parameters k, a and b are defined in [12]. It should be noted that the C_λ-extended oscillator algebra worked out in [15] is a particular case of W_k (for $\lambda = k$).

2.2. Hilbertian representation of $\mathcal{A}_\kappa$

We denote by $\mathcal{F}_\kappa$ the finite- or infinite-dimensional Hilbert space on which the operators a^-, a^+ and N are defined. Let

$$\{|n\rangle : n = 0, 1, \ldots, d(\kappa)\} \tag{3}$$

(with $d(\kappa)$ being finite or infinite) be an orthonormal basis, with respect to the inner product $\langle n|n'\rangle = \delta_{n,n'}$, of the space $\mathcal{F}_\kappa$. It is easy to check that the actions

$$
\begin{aligned}
a^+|n\rangle &= \sqrt{F(n+1)}\, \mathrm{e}^{-\mathrm{i}[F(n+1)-F(n)]\varphi}|n+1\rangle, \\
a^-|n\rangle &= \sqrt{F(n)}\, \mathrm{e}^{\mathrm{i}[F(n)-F(n-1)]\varphi}|n-1\rangle, \\
a^-|0\rangle &= 0 \qquad N|n\rangle = n|n\rangle
\end{aligned}
\tag{4}
$$

provide a Hilbertian representation of the algebra $\mathcal{A}_\kappa$ defined by (1). In equation (4), the real parameter φ is arbitrary, and the positively valued function $F : \mathbb{N} \to \mathbb{R}_+$ satisfies the recurrence relation

$$F(n+1) - F(n) = 1 + 2\kappa n \qquad F(0) = 0. \tag{5}$$

The iteration of (5) yields

$$F(n) = n[1 + \kappa(n-1)], \tag{6}$$

which is linear in n only for $\kappa = 0$. Since $F(n) \in \mathbb{R}_+$, we must have the following condition:

$$1 + \kappa(n-1) > 0 \tag{7}$$

for $n > 0$. Condition (7) determines the value of $d(\kappa)$ and then the dimension of $\mathcal{F}_\kappa$. The finiteness or infiniteness of $\mathcal{F}_\kappa$ depends on the sign of the parameter κ. For $\kappa \geqslant 0$, the space $\mathcal{F}_\kappa$ is infinite dimensional. In fact, for $\kappa = 0$, the space $\mathcal{F}_0$ coincides with the usual Hilbert–Foch space for the harmonic oscillator. For $\kappa < 0$, there exists a finite number of states satisfying condition (7). As a matter of fact, for $\kappa < 0$, n can take the values

$$n = 0, 1, \ldots, E\left(-\frac{1}{\kappa}\right) \equiv d - 1, \tag{8}$$

where $E(x)$ stands for the integer part of x. The finiteness of the space $\mathcal{F}_\kappa$ induces properties of the operators a^- and a^+ which differ from those corresponding to an infinite-dimensional space. In particular, the trace of any commutator in the finite-dimensional space must be zero. This implies that the parameter κ is related to the dimension d of the space $\mathcal{F}_\kappa$ by

$$d = 1 - \frac{1}{\kappa}. \tag{9}$$

Equation (9) requires that $-1/\kappa$ be a positive integer. In the following, we shall assume that $-1/\kappa \in \mathbb{N}^*$ when $\kappa < 0$.

2.3. A generalized oscillator Hamiltonian

We are now in a position to define an operator which generalizes (up to an additive constant) the Hamiltonian $a^+ a^- + 1/2$ for the one-dimensional harmonic oscillator. Starting from

$$a^+ a^- |n\rangle = F(n)|n\rangle \Rightarrow F(N) = a^+ a^-, \tag{10}$$

we refer $F(N)$ to as an Hamiltonian associated with the generalized oscillator algebra $\mathcal{A}_\kappa$. The eigenvalue equation

$$F(N)|n\rangle = n[1 + \kappa(n-1)]|n\rangle \tag{11}$$

gives the energies (6) of a quantum dynamical system described by the Hamiltonian operator $F(N)$. Let us discuss the degeneracies of the levels $F(n)$ given by (6).

  (i) In the case $\kappa \geqslant 0$, the spectrum of $F(N)$ is nondegenerate.
 (ii) In the case $\kappa < 0$, the eigenvalues of $F(N)$ can be rewritten as

$$F(n) = n\frac{d-n}{d-1}, \tag{12}$$

so that

$$F(n) = F(d-n) \qquad n = 1, 2, \ldots, d-1. \tag{13}$$

Thus, for d even, the levels are doublets except the fundamental level $n = 0$ and the level $n = d/2$ which are nondegenerate. For d odd, the levels are two-fold degenerate except the fundamental level $n = 0$ which is a singlet.

In both cases ($\kappa \geqslant 0$ and $\kappa < 0$), we note that the Perron–Frobenius theorem [16] is satisfied, namely, the fundamental level is nondegenerate.

It is known that one-dimensional quantum dynamical systems (on the real line) correspond to nondegenerate spectra. Therefore, the representation obtained for $\mathcal{A}_\kappa$ with $\kappa < 0$ cannot be used to describe a particle evolving in some nonrelativistic potential on the real line. However, a modification of the generalized oscillator algebra $\mathcal{A}_\kappa$ can be achieved in order to avoid the degeneracies of $F(N)$. This will be done in section 4 by means of a truncation procedure which will prove also to be useful in the case $\kappa \geqslant 0$ to define in a consistent way the phase operator for some exactly solvable systems.

3. Temporally stable phase states for $\mathcal{A}_\kappa$

We shall treat separately the cases $\kappa \geqslant 0$ and $\kappa < 0$ associated with the infinite- and the finite-dimensional representation of the generalized oscillator algebra $\mathcal{A}_\kappa$, respectively.

3.1. The infinite-dimensional case

In the case $\kappa \geqslant 0$, we decompose a^- and a^+ as

$$a^- = E_\infty \sqrt{F(N)} \qquad a^+ = \sqrt{F(N)}\,(E_\infty)^\dagger, \tag{14}$$

where

$$E_\infty := \sum_{n=0}^\infty e^{i[F(n+1)-F(n)]\varphi}|n\rangle\langle n+1|. \tag{15}$$

It is important to emphasize that

$$E_\infty\,(E_\infty)^\dagger = \sum_{n=0}^\infty |n\rangle\langle n| = I \qquad (E_\infty)^\dagger E_\infty = \sum_{n=1}^\infty |n\rangle\langle n| = I - |0\rangle\langle 0|, \tag{16}$$

a result which means that E_∞ is not a unitary operator.

To find the phase states corresponding to $\kappa \geqslant 0$, let us consider the eigenvalue equation

$$E_\infty|z\rangle = z|z\rangle \qquad z \in \mathbb{C}. \tag{17}$$

By expanding the vector $|z\rangle$ of $\mathcal{F}_\kappa$ as

$$|z\rangle = \sum_{n=0}^{\infty} C_n z^n |n\rangle, \tag{18}$$

it is easy to see that the complex coefficients C_n satisfy the relation

$$C_{n+1} = \mathrm{e}^{-\mathrm{i}[F(n+1)-F(n)]\varphi} C_n \qquad n \in \mathbb{N}. \tag{19}$$

It follows that

$$C_n = \mathrm{e}^{-\mathrm{i}F(n)\varphi} C_0 \qquad n \in \mathbb{N}, \tag{20}$$

where the coefficient C_0 can be determined from the normalization condition of the states $|z\rangle$. As a result, we can take (up to a phase factor)

$$|z\rangle = \sqrt{1 - |z|^2} \sum_{n=0}^{\infty} z^n \, \mathrm{e}^{-\mathrm{i}F(n)\varphi} |n\rangle \tag{21}$$

on the domain $\{z \in \mathbb{C}, |z| < 1\}$.

Following the method developed in [17] for the Lie algebra $su(1, 1)$, we define the states $|\theta, \varphi\rangle$ by

$$|\theta, \varphi\rangle := \lim_{z \to \mathrm{e}^{\mathrm{i}\theta}} \frac{1}{\sqrt{1 - |z|^2}} |z\rangle, \tag{22}$$

where $\theta \in [-\pi, +\pi]$ (see also [18] where a limit of type $z \to \mathrm{e}^{\mathrm{i}\theta} \Rightarrow |z| \to 1$ is used in a similar way). We thus get the states

$$|\theta, \varphi\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n\theta} \, \mathrm{e}^{-\mathrm{i}F(n)\varphi} |n\rangle. \tag{23}$$

These states, defined on the unit circle S^1, turn out to be phase states. Indeed, we have

$$E_\infty|\theta, \varphi\rangle = \mathrm{e}^{\mathrm{i}\theta}|\theta, \varphi\rangle. \tag{24}$$

Hence, the operator E_∞ is a (nonunitary) phase operator.

The main properties of the states $|\theta, \varphi\rangle$ are the following.

(i) They are temporally stable in the sense that the relation

$$\mathrm{e}^{-\mathrm{i}F(N)t}|\theta, \varphi\rangle = |\theta, \varphi + t\rangle \tag{25}$$

is satisfied for any value of the real parameter t. This property is due to the presence of the parameter φ in the phase operator E_∞.

(ii) They are not normalized and not orthogonal. However, for fixed φ, they satisfy the closure relation

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta |\theta, \varphi\rangle\langle\theta, \varphi| = I. \tag{26}$$

Finally, observe that for $\varphi = 0$ the states $|\theta, 0\rangle$ have the same form as those derived in [17] for $su(1, 1)$.

3.2. The finite-dimensional case

For $\kappa < 0$ with $-1/\kappa \in \mathbb{N}^*$, the Hilbert space $\mathcal{F}_\kappa$ is d dimensional with $d = 1 - 1/\kappa$. The action of a^- and a^+ on $\mathcal{F}_\kappa$ is given by (4) supplemented by

$$a^+|d - 1\rangle = 0, \tag{27}$$

which easily follows from the calculation of $\langle d - 1|a^- a^+|d - 1\rangle$.

Let us look for a decomposition of the creation a^+ and annihilation a^- operators similar to (14) for the case $\kappa \geqslant 0$. Thus, let us put

$$a^- = E_d\sqrt{F(N)} \Leftrightarrow a^+ = \sqrt{F(N)}\,(E_d)^\dagger. \tag{28}$$

The operator E_d can be seen to satisfy

$$E_d|n\rangle = \mathrm{e}^{\mathrm{i}[F(n)-F(n-1)]\varphi}|n - 1\rangle \tag{29}$$

for $n = 1, 2, \ldots, d - 1$. For $n = 0$, we shall assume that

$$E_d|0\rangle = \mathrm{e}^{\mathrm{i}[F(0)-F(d-1)]\varphi}|d - 1\rangle, \tag{30}$$

so that (29) is valid modulo d. (Note that, in view of (28), $a^-|0\rangle = 0$ does not imply that $E_d|0\rangle = 0$.) It follows that we have

$$(E_d)^\dagger|n\rangle = \mathrm{e}^{-\mathrm{i}[F(n+1)-F(n)]\varphi}|n + 1\rangle, \tag{31}$$

where $n + 1$ should be understood modulo d. As an important result (to be contrasted with the situation where $\kappa \geqslant 0$), the operator E_d is unitary. Therefore, equation (28) constitutes a polar decomposition of a^- and a^+.

We are now ready to derive the eigenstates of the operator E_d. Let us consider the eigenvalue equation

$$E_d|z\rangle = z|z\rangle \qquad |z\rangle = \sum_{n=0}^{d-1} C_n z^n|n\rangle \tag{32}$$

with $z \in \mathbb{C}$. Here again (as in the case $\kappa \geqslant 0$), we obtain a recurrence relation for the coefficients C_n, namely,

$$C_n = \mathrm{e}^{-\mathrm{i}[F(n)-F(n-1)]\varphi}C_{n-1} \qquad n = 1, 2, \ldots, d - 1, \tag{33}$$

with the cyclic condition

$$C_0 = z^d\,\mathrm{e}^{-\mathrm{i}[F(0)-F(d-1)]\varphi}C_{d-1}. \tag{34}$$

Therefore, we get

$$C_n = \mathrm{e}^{-\mathrm{i}F(n)\varphi}C_0 \qquad n = 0, 1, \ldots, d - 1, \tag{35}$$

with the discretization condition

$$z^d = 1. \tag{36}$$

As a consequence, the complex variable z is a root of unity given by

$$z = q^m \qquad m = 0, 1, \ldots, d - 1, \tag{37}$$

where

$$q := \mathrm{e}^{2\pi\mathrm{i}/d} \tag{38}$$

is reminiscent of the parameter used in the theory of quantum groups. The constant C_0 can be calculated from the normalization condition $\langle z|z\rangle = 1$ to be

$$C_0 = \frac{1}{\sqrt{d}} \tag{39}$$

up to a phase factor. Finally, we arrive at the following eigenstates $|z\rangle \equiv |m, \varphi\rangle$ of E_d:

$$|m, \varphi\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \mathrm{e}^{-\mathrm{i}F(n)\varphi} q^{mn} |n\rangle. \tag{40}$$

The states $|m, \varphi\rangle$, labeled by the parameters $m \in \mathbb{Z}/d\mathbb{Z}$ and $\varphi \in \mathbb{R}$, satisfy

$$E_d|m, \varphi\rangle = \mathrm{e}^{\mathrm{i}\theta_m}|m, \varphi\rangle \qquad \theta_m = m\frac{2\pi}{d}, \tag{41}$$

which shows that E_d is indeed a phase operator. In the particular case $\varphi = 0$, the states $|m, 0\rangle$ are similar to those derived in [17] for the Lie algebra $su(2)$. In this case, the states $|m, 0\rangle$ correspond to an ordinary discrete Fourier transform of the basis $\{|n\rangle : n = 0, 1, \ldots, d-1\}$ of the d-dimensional space $\mathcal{F}_\kappa$.

The phase states $|m, \varphi\rangle$ have remarkable properties (to be compared to those for the states $|\theta, \varphi\rangle$ of the case $\kappa \geqslant 0$).

(i) They are temporally stable under 'time evolution'. In other words, they satisfy

$$\mathrm{e}^{-\mathrm{i}F(N)t}|m, \varphi\rangle = |m, \varphi + t\rangle. \tag{42}$$

for any value of the real parameter t. We note here the major role of the parameter φ in ensuing the temporal stability of the states $|m, \varphi\rangle$.

(ii) For fixed φ, they satisfy the equiprobability relation

$$|\langle n|m, \varphi\rangle| = \frac{1}{\sqrt{d}} \qquad n, m \in \mathbb{Z}/d\mathbb{Z}. \tag{43}$$

(iii) For fixed φ, they satisfy the orthonormality relation

$$\langle m, \varphi|m', \varphi\rangle = \delta_{m,m'} \qquad m, m' \in \mathbb{Z}/d\mathbb{Z} \tag{44}$$

and the closure property

$$\sum_{m=0}^{d-1} |m, \varphi\rangle\langle m, \varphi| = I. \tag{45}$$

(iv) The overlap between two phase states $|m', \varphi'\rangle$ and $|m, \varphi\rangle$ reads

$$\langle m, \varphi|m', \varphi'\rangle = \frac{1}{d} \sum_{n=0}^{d-1} q^{\rho(m-m', \varphi-\varphi', n)}, \tag{46}$$

where

$$\rho(m - m', \varphi - \varphi', n) = -(m - m')n + \frac{d}{2\pi}(\varphi - \varphi')F(n) \tag{47}$$

and q is defined in (38). Therefore, the temporally stable phase states are not all orthogonal.

4. Truncated generalized oscillator algebra and phase states

As discussed in section 2, in the case $\kappa \geqslant 0$ the Hilbert space $\mathcal{F}_\kappa$ associated with $\mathcal{A}_\kappa$ is infinite dimensional. It is then impossible to define a unitary phase operator (see section 3). On the other hand, in the case $\kappa < 0$ with $-1/\kappa \in \mathbb{N}^*$ the space $\mathcal{F}_\kappa$ is finite dimensional, and there is no problem to define a unitary phase operator. However, the spectrum of the Hamiltonian $F(N)$ associated with $\mathcal{A}_\kappa$ for $-1/\kappa \in \mathbb{N}^*$ exhibits degeneracies. Therefore, it is appropriate to truncate the space $\mathcal{F}_\kappa$ for both $\kappa \geqslant 0$ and $\kappa < 0$ in order to get a subspace $\mathcal{F}_{\kappa,s}$ of dimension s with the basis $\{|n\rangle : n = 0, 1, \ldots, s-1\}$. For $\kappa \geqslant 0$, the truncation is done at s sufficiently large (note that the difference $F(n+1) - F(n)$ between two consecutive states increases with n for $\kappa > 0$ so that we can ignore, in a perturbative scheme, the states with n large). For $\kappa < 0$, the truncation can be done at $s = (d+2)/2$ for d even and at $s = (d+1)/2$ for d odd (with d given by (9)) in order to avoid the degeneracies of $F(N)$.

4.1. The truncated algebra $\mathcal{A}_{\kappa,s}$

Inspired by the work of Pegg and Barnett [4], we define the truncated generalized oscillator algebra $\mathcal{A}_{\kappa,s}$ through the three linear operators b^-, b^+ and N satisfying the following relations:

$$[b^-, b^+] = I + 2\kappa N - F(s)|s-1\rangle\langle s-1| \quad [N, b^\pm] = \pm b^\pm \quad (b^-)^\dagger = b^+ \quad N^\dagger = N. \quad (48)$$

The algebra $\mathcal{A}_{\kappa,s}$ generalizes the one introduced by Pegg and Barnett for the harmonic oscillator in their discussion of the phase operator for the single modes of the electromagnetic field [4]. Indeed, the algebra $\mathcal{A}_{0,s}$, for $\kappa = 0$, is identical to the truncated oscillator algebra considered in [4].

Following the same approach as in subsection 2.2, we define an s-dimensional representation of $\mathcal{A}_{\kappa,s}$ (whatever the sign of κ is) via the actions

$$
\begin{aligned}
b^+|n\rangle &= \sqrt{F(n+1)}\, e^{-i[F(n+1)-F(n)]\varphi}|n+1\rangle, \\
b^-|n\rangle &= \sqrt{F(n)}\, e^{i[F(n)-F(n-1)]\varphi}|n-1\rangle, \\
b^-|0\rangle &= 0 \qquad b^+|s-1\rangle = 0 \qquad N|n\rangle = n|n\rangle
\end{aligned}
\quad (49)
$$

for $n = 0, 1, \ldots, s-1$. Note that a further condition is necessary here, namely, the upper limit condition $b^+|s-1\rangle = 0$. It can be checked that the recurrence relation (5) is equally valid for $\mathcal{A}_{\kappa,s}$. Therefore, equations (6) and (12) can be applied with $n = 0, 1, \ldots, s-1$.

It is interesting to note that the creation and annihilation operators b^- and b^+ satisfy (in the representation under consideration) the nilpotency relations

$$(b^-)^s = (b^+)^s = 0, \quad (50)$$

which are similar (for $s = k \in \mathbb{N} \setminus 0, 1$) to those describing the so-called k-fermions that are objects interpolating between fermions (for $k = 2$) and bosons (for $k \to \infty$) [19].

4.2. Phase states for $\mathcal{A}_{\kappa,s}$

For the truncated algebra $\mathcal{A}_{\kappa,s}$ (corresponding to $d(\kappa)$ finite or infinite), the analog of the phase operator E_d is the unitary operator

$$E_s := e^{i[F(0)-F(s-1)]\varphi}|s-1\rangle\langle 0| + \sum_{n=1}^{s-1} e^{i[F(n)-F(n-1)]\varphi}|n-1\rangle\langle n|. \quad (51)$$

By using the same reasoning as in subsection 3.2, we obtain

$$E_s|m,\varphi\rangle = e^{i\theta_m}|m,\varphi\rangle \qquad \theta_m = m\frac{2\pi}{s}, \quad (52)$$

where

$$|m,\varphi\rangle = \frac{1}{\sqrt{s}} \sum_{n=0}^{s-1} e^{-iF(n)\varphi}(q_s)^{mn}|n\rangle, \quad (53)$$

with $m \in \mathbb{Z}/s\mathbb{Z}$, $\varphi \in \mathbb{R}$ and q_s given by

$$q_s := e^{2\pi i/s}. \quad (54)$$

We are thus left with phase states $|m,\varphi\rangle$ associated with the phase operator E_s. These states satisfy the same properties as those for E_d (see section 3.2) except that d is replaced by s in some places.

4.3. A new type of discrete phase states

It is well known that, for quantum systems with a finite spectrum (like the Morse system) or for Lie algebras with finite-dimensional unitary representations (as for instance $su(2)$), the construction of coherent states cannot be achieved by looking for the eigenstates of an annihilation operator [8] or of a compact shift operator [6].

For the algebra $\mathcal{A}_{\kappa,s}$, the difficulty inherent to the finiteness of the representation can be overcome as follows. We define the operator

$$V_s := b^- + \frac{(b^+)^{s-1}}{E(s-1)}, \tag{55}$$

where the function E is defined via

$$E(0) := 1 \qquad E(n) := F(1)F(2)\cdots F(n) \qquad n = 1, 2, \ldots, s-1. \tag{56}$$

The operator V_s is an idempotent operator of order s since

$$(V_s)^s = I. \tag{57}$$

Let us consider the eigenvalue equation

$$V_s|z\rangle = z|z\rangle \qquad |z\rangle = \sum_{n=0}^{s-1} C_n z^n |n\rangle \qquad z \in \mathbb{C}. \tag{58}$$

By using (57), we obtain that z is discretized as

$$z = (q_s)^\mu \qquad \mu \in \mathbb{Z}/s\mathbb{Z}, \tag{59}$$

with q_s being defined by (54). Then, it is a simple matter to calculate the coefficients C_n and to normalize the μ- and φ-dependent states $|z\rangle \equiv |\mu, \varphi\rangle$. This leads to

$$|\mu, \varphi\rangle = C_0 \sum_{n=0}^{s-1} \frac{1}{\sqrt{E(n)}} (q_s)^{n\mu} \, e^{-iF(n)\varphi} |n\rangle, \tag{60}$$

where the normalization factor C_0 is such that (up to a phase factor)

$$C_0^{-2} = \sum_{n=0}^{s-1} \frac{1}{E(n)}. \tag{61}$$

The states $|\mu, \varphi\rangle$ are temporally stable and are similar to the coherent states introduced by Gazeau and Klauder [8] except that their labeling includes an integer and they correspond to the eigenvectors of a polynomial in terms of generalized creation and annihilation operators. They satisfy

$$\langle \mu, \varphi | \mu', \varphi' \rangle = C_0^2 \sum_{n=0}^{s-1} \frac{1}{E(n)} (q_s)^{n(\mu'-\mu)} \, e^{-iF(n)(\varphi'-\varphi)} \tag{62}$$

and

$$\frac{1}{s} \sum_{m=0}^{s-1} |\mu, \varphi\rangle\langle\mu, \varphi| = C_0^2 \sum_{n=0}^{s-1} \frac{1}{E(n)} |n\rangle\langle n|. \tag{63}$$

Consequently, they are not orthogonal.

We close this subsection with a remark concerning the unitary operator

$$U_s := (q_s)^N \tag{64}$$

that is a companion of V_s in the following sense. This operator satisfies the cyclicity relation

$$(U_s)^s = I. \tag{65}$$

Furthermore, we have the s-commutation relation

$$V_s U_s - q_s U_s V_s = 0. \tag{66}$$

Equations (57), (65) and (66) are necessary conditions for the pair (U_s, V_s) be a pair of Weyl (see [14]). However, this is not the case because V_s is not unitary.

5. Application to mutually unbiased bases

As an *a priori* unexpected connection, the approach in subsection 3.2 and 4.2 for the finite-dimensional cases (for $\mathcal{A}_\kappa$ and $\mathcal{A}_{\kappa,s}$) can be further developed for deriving MUBs. Let us recall that two orthonormal bases $\{|a\alpha\rangle : \alpha = 0, 1, \ldots, d-1\}$ and $\{|b\beta\rangle : \beta = 0, 1, \ldots, d-1\}$ in a d-dimensional Hilbert space (with an inner product $\langle\,|\,\rangle$) are said to be mutually unbiased iff

$$|\langle a\alpha|b\beta\rangle| = \delta_{a,b}\delta_{\alpha,\beta} + \frac{1}{\sqrt{d}}(1 - \delta_{a,b}). \tag{67}$$

For fixed d, it is known that the number $\mathcal{N}$ of MUBs is such that $\mathcal{N} \leqslant d+1$ and that the limit $\mathcal{N} = d+1$ is reached when d is the power of a prime number [20, 21].

5.1. MUBs from phase states for $\mathcal{A}_\kappa$

In order to generate MUBs along the line of the developments of subsection 3.2, let us further examine some properties of the phase operator E_d for $\mathcal{A}_\kappa$ with $\kappa < 0$. This operator can be written in a compact form as

$$E_d = \sum_{n=0}^{d-1} e^{i[F(n)-F(n-1)]\varphi}|n-1\rangle\langle n| \tag{68}$$

(in this section, the summations on n are understood modulo d). It is easy to check that

$$(E_d)^d = I, \tag{69}$$

so that E_d is idempotent. The operator E_d can be decomposed as

$$E_d = U_\varphi V, \tag{70}$$

where the operators U_φ and V are defined by

$$U_\varphi := e^{i[F(N+1)-F(N)]\varphi} \qquad V := \sum_{n=0}^{d-1}|n-1\rangle\langle n|. \tag{71}$$

The operators U_φ and V are unitary and satisfy the pseudocommutation relation

$$U_\varphi V = e^{2i\varphi/(d-1)}V U_\varphi. \tag{72}$$

In addition, the operator V satisfies the idempotency relation

$$V^d = I \tag{73}$$

and, when the parameter φ is quantized as

$$\varphi = -\pi\frac{d-1}{d}p \qquad p \in \mathbb{Z}/d\mathbb{Z}, \tag{74}$$

we have

$$(U_\varphi)^d = e^{i\pi(d-1)p}I. \tag{75}$$

10

In view of (74), equation (72) can be rewritten as

$$VU_\varphi = q^p U_\varphi V \qquad (76)$$

(see (38) for the definition of q). For the discrete values of φ afforded by (74), equation (40) yields the phase states $|m, \varphi\rangle \equiv |m, p\rangle$ given by

$$|m, p\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} q^{n(d-n)p/2+nm} |n\rangle \qquad p, m \in \mathbb{Z}/d\mathbb{Z}, \qquad (77)$$

which coincides with the vector $|a\alpha\rangle$, with $a \equiv p$ and $\alpha \equiv m$, obtained in [13] in an $SU(2)$ approach to MUBs. Alternatively, by putting

$$k := d - n - 1 \qquad |n\rangle = |d - k - 1\rangle \equiv |k\rangle, \qquad (78)$$

equation (77) becomes

$$|m, p\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} q^{(k+1)(d-k-1)p/2-(k+1)m} |k\rangle, \qquad p, m \in \mathbb{Z}/d\mathbb{Z}, \qquad (79)$$

which coincides with the vector $|a\alpha\rangle$, with $a \equiv p$ and $\alpha \equiv m$, derived in [14] in an angular momentum approach to MUBs. It is to be observed that (77) and (79) correspond to quadratic discrete Fourier transforms.

To make a further contact with [13, 14], let us note that when φ is discretized according to (74), the inner product $\langle m, \varphi|m', \varphi'\rangle \equiv \langle m, p|m', p'\rangle$ (see equation (46)) can be rewritten as

$$\langle m, p|m', p'\rangle = \frac{1}{d} S(u, v, w) \qquad (80)$$

with

$$u := p - p' \qquad v := -(p - p')d + 2(m' - m) \qquad w := d. \qquad (81)$$

In equation (80), the factor $S(u, v, w)$ denotes a generalized quadratic Gauss sum defined by [22]

$$S(u, v, w) := \sum_{k=0}^{|w|-1} e^{i\pi(uk^2+vk)/w}, \qquad (82)$$

where u, v and w are integers (the nonvanishing of $S(u, v, w)$ requires $uw + v$ even). In the special case where d is a prime integer and $p' \neq p$, the calculation of $S(u, v, w)$ in (80) through the methods developed in [22, 23] (see also [13]) leads to

$$|\langle m, p|m', p'\rangle| = \frac{1}{\sqrt{d}}. \qquad (83)$$

This result shows that the d bases

$$B_p := \{|m, p\rangle : m = 0, 1, \ldots, d-1\} \qquad p = 0, 1, \ldots, d-1 \qquad (84)$$

of the d-dimensional space $\mathcal{F}_\kappa$, with d given by (9) are mutually unbiased. On the other hand, in view of (43), it is clear that any basis B_p and the basis

$$B_d := \{|n\rangle : n = 0, 1, \ldots, d-1\}, \qquad (85)$$

known as the computational basis in quantum information and quantum computation, are mutually unbiased. As a conclusion, for d prime, the d bases B_p with $p = 0, 1, \ldots, d-1$ and the computational basis B_d constitute a complete set of $d+1$ MUBs. This result, in agreement with the one derived in [13, 14], is the starting point for constructing MUBs in power prime dimension.

5.2. MUBs from phase states for $\mathcal{A}_{\kappa,s}$

By applying a discretization procedure similar to the one introduced in subsection 5.1, we can construct MUBs from the phase states (53) for the truncated algebra $\mathcal{A}_{\kappa,s}$, with $\kappa \neq 0$. Let us quantize the parameter φ by putting

$$\varphi = \pi \frac{2}{s\kappa} p \qquad p \in \mathbb{Z}/s\mathbb{Z}. \tag{86}$$

Then, equation (53) leads to the states $|m, \varphi\rangle \equiv |m, p\rangle$ given by

$$|m, p\rangle = \frac{1}{\sqrt{s}} \sum_{n=0}^{s-1} (q_s)^{n(\delta-n)p+nm} |n\rangle \qquad p, m \in \mathbb{Z}/s\mathbb{Z}, \tag{87}$$

where

$$\delta := 1 - \frac{1}{\kappa}. \tag{88}$$

In this subsection, we shall assume that $1/\kappa \in \mathbb{Z}$ so that $\delta \in \mathbb{Z}$ (note that $\delta = d$ for $\kappa < 0$). The overlap $\langle m, p | m', p'\rangle$ can be written in terms of the generalized quadratic Gauss sum (82). Indeed, we have

$$\langle m, p | m', p'\rangle = \frac{1}{s} \sum_{n=0}^{s-1} (q_s)^{n(\delta-n)(p'-p)+n(m'-m)} = \frac{1}{s} S(u, v, w), \tag{89}$$

where

$$u := 2(p - p') \qquad v := 2\delta(p' - p) + 2(m' - m) \qquad w := s. \tag{90}$$

We can proceed as in subsection 5.1 in order to show that the various states $|m, p\rangle$ generate, together with the s-dimensional basis $\{|n\rangle : n = 0, 1, \ldots, s - 1\}$, $s + 1$ MUBs when s is a prime integer.

6. Application to exactly solvable potentials

The main goal of this section is to show how the generalized oscillator algebra $\mathcal{A}_\kappa$ is relevant for the study of one-dimensional exactly solvable potentials in the context of supersymmetric quantum mechanics and how MUBs can be derived from the temporally stable phase states for some quantum mechanical systems.

6.1. Creation, annihilation and transfer operators

Ordinary supersymmetric quantum mechanics can be presented in different ways (e.g. see [24–30]). We adopt here the approach according to which a supersymmetric dynamical system is defined by a triplet $(H, Q_+, Q_-)_2$ of linear operators acting on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ and satisfying the following relations:

$$H = H^\dagger \qquad Q_- = Q_+^\dagger \qquad Q_\pm^2 = 0 \qquad \{Q_-, Q_+\} = H \qquad [H, Q_\pm] = 0. \tag{91}$$

(In this approach, ordinary supersymmetric quantum mechanics is a particular case, corresponding to $k = 2$, of fractional supersymmetric quantum mechanics of order k dealing with triplets $(H, Q_+, Q_-)_k$ which satisfy relations generalizing (91) and which correspond to a $\mathbb{Z}_k$ grading [12].) The operators Q_+ and Q_- are the supercharges of the one-dimensional system. We suppose that the spectrum of the self-adjoint operator H, the supersymmetric Hamiltonian of the system, is discrete. The Hamiltonian H can be written as

$$H = H_0 + H_1, \tag{92}$$

where H_0 and H_1 act on the states $|\Psi_n, 0\rangle$ and $|\Psi_n, 1\rangle$ of even and odd grading, respectively. In other words, the Hilbert space $\mathcal{H}$ is decomposed as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \qquad \mathcal{H}_0 := \{|\Psi_n, 0\rangle : n \text{ ranging}\} \qquad \mathcal{H}_1 := \{|\Psi_n, 1\rangle : n \text{ ranging}\}, \qquad (93)$$

which reflects the $\mathbb{Z}_2$ grading. We shall assume that there is no supersymmetry breaking. In this case, the Hamiltonians H_0 and H_1 are isospectral except that the ground state of H_0 has no supersymmetric partner in the spectrum of H_1.

By combining the above-mentioned considerations on supersymmetry with the Infeld and Hull factorization method [28], we can construct creation, annihilation and transfer operators for an exactly solvable Hamiltonian in one dimension [24–35]. For this purpose, let us consider a one-dimensional quantum system embedded in a real potential $v_0 : x \mapsto v_0(x)$. The corresponding Hamiltonian is

$$H_0 := -\frac{1}{2}\frac{d^2}{dx^2} + v_0. \qquad (94)$$

Let us suppose that the Hamiltonian H_0 is exactly solvable and admits the discrete spectrum

$$e_0 = 0 < e_1 < e_2 < \cdots < e_n < e_{n+1} < \cdots, \qquad (95)$$

with a finite or infinite number of levels. We know that the Hamiltonian H_0 of this system can be factorized as [25, 26, 28–30]

$$H_0 = x^+ x^- \qquad x^+ := \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + w\right) \qquad x^- := \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + w\right). \qquad (96)$$

The superpotential $w : x \mapsto w(x)$ satisfies the Ricatti equation

$$v_0 = \frac{1}{2}\left(w^2 - \frac{\mathrm{d}w}{\mathrm{d}x}\right). \qquad (97)$$

Since the ground state energy is assumed to be zero, it is easy to see that the potential v_0 and the superpotential w can be expressed in terms of the ground state wavefunction.

It is important to stress that the operators x^+ and x^- are not in general creation and annihilation operators for H_0 [25, 26, 29, 30, 35]. They are indeed transfer operators from the spectrum of H_0 to the one of the H_1 and vice versa. To identify them, we start by representing the supercharge operators and the supersymmetric Hamiltonian by 2×2 matrices [25–27, 30]

$$Q_- = \begin{pmatrix} 0 & x^- \\ 0 & 0 \end{pmatrix} \qquad Q_+ = \begin{pmatrix} 0 & 0 \\ x^+ & 0 \end{pmatrix} \qquad H = \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix}, \qquad (98)$$

where the operator

$$H_1 = x^- x^+ = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v_1 \qquad (99)$$

is the supersymmetric partner of H_0 and corresponds to a new potential $v_1 : x \mapsto v_1(x)$. The potential

$$v_1 = \frac{1}{2}\left(w^2 + \frac{\mathrm{d}w}{\mathrm{d}x}\right) \qquad (100)$$

is the supersymmetric partner of the potential v_0. The Hamiltonian H_1 is also exactly solvable and isospectral to H_0 (except for the ground state). Indeed,

$$H_0|\Psi_n, 0\rangle = e_n^0|\Psi_n, 0\rangle \Rightarrow H_1(x^-|\Psi_n, 0\rangle) = e_n^0(x^-|\Psi_n, 0\rangle), \qquad (101)$$

where $e_n^0 := e_n$. Similarly,

$$H_1|\Psi_n, 1\rangle = e_n^1|\Psi_n, 1\rangle \Rightarrow H_0(x^+|\Psi_n, 1\rangle) = e_n^1(x^+|\Psi_n, 1\rangle). \qquad (102)$$

(For more details, see [25, 26, 29] and the recent topical review [30].) From equations (101) and (102), it is clear that we can take

$$x^-|\Psi_{n+1}, 0\rangle = \sqrt{e_{n+1}}\, e^{i(e_{n+1}-e_n)\varphi}|\Psi_n, 1\rangle \tag{103}$$

$$x^+|\Psi_n, 1\rangle = \sqrt{e_{n+1}}\, e^{-i(e_{n+1}-e_n)\varphi}|\Psi_{n+1}, 0\rangle, \tag{104}$$

where φ is a real number, and that the energies of the supersymmetric partners H_0 and H_1 are related by

$$e_n^1 = e_{n+1}^0. \tag{105}$$

Note that the operator x^- (respectively x^+) converts an eigenfunction of H_0 (respectively H_1) into an eigenfunction of H_1 (respectively H_0) with the same energy. Thus, the operators x^- and x^+ transfer the states from one spectrum to its partner spectrum. To introduce the ladder operators inside a given spectrum, we first consider the unitary operator U relating the states $|\Psi_n, 0\rangle$ and $|\Psi_n, 1\rangle$ through (cf [31–33])

$$U := \sum_n |\Psi_n, 1\rangle\langle\Psi_n, 0| \Rightarrow |\Psi_n, 1\rangle = U|\Psi_n, 0\rangle. \tag{106}$$

Operators similar to U were already considered for continuous spectra [31, 33] and for discrete spectra [32, 34]. Then, we define the operators [32–34]

$$a^+ := x^+U \qquad a^- := U^\dagger x^-. \tag{107}$$

By using equations (103) and (104), we obtain

$$a^-|\Psi_n, 0\rangle = \sqrt{e_n}\, e^{i(e_n-e_{n-1})\varphi}|\Psi_{n-1}, 0\rangle \tag{108}$$

$$a^+|\Psi_n, 0\rangle = \sqrt{e_{n+1}}\, e^{-i(e_{n+1}-e_n)\varphi}|\Psi_{n+1}, 0\rangle. \tag{109}$$

Consequently, a^+ and a^- are creation and annihilation operators for the Hamiltonian H_0. Furthermore, it is easily seen that

$$a^+a^- = x^+x^- = H_0. \tag{110}$$

Ladder operators for the Hamiltonian H_1 can be introduced in a similar way.

6.2. Physical realizations of the generalized oscillator algebra

To simplify the notation, we set $|\Psi_n\rangle := |\Psi_n, 0\rangle$. From equations (108) and (109), we get

$$[a^-, a^+]|\Psi_n\rangle = (e_{n+1} - e_n)|\Psi_n\rangle. \tag{111}$$

The number operator N defined by

$$N|\Psi_n\rangle = n|\Psi_n\rangle \tag{112}$$

is in general (for an arbitrary quantum system) different from the product a^+a^-. Let us consider the situation where the creation and annihilation operators satisfy the commutation relation

$$[a^-, a^+] = aN + b, \tag{113}$$

a relation used in the study of the so-called polynomial Heisenberg algebra introduced in [36]. In other words, we assume that the energy gap $e_{n+1} - e_n$ between two successive levels is linear in n, i.e.

$$e_{n+1} - e_n = an + b, \tag{114}$$

14

where a and b are two real parameters. We also assume that the eigenvalues of the operator $aN + b$ are positive. With these choices, the algebra generated by the operators a^+, a^- and N is identical to the generalized oscillator algebra $\mathcal{A}_\kappa$ modulo the replacements

$$a^\pm \rightarrow \frac{a^\pm}{\sqrt{b}} \qquad \kappa \rightarrow \frac{1}{2}\frac{a}{b} \tag{115}$$

in equation (1). Thus, from equations (108)–(110), we have

$$H_0 = a^+a^- = \tfrac{1}{2}aN(N-1) + bN. \tag{116}$$

For $a \neq 0$, the spectrum of H_0 is nonlinear and is given by

$$H_0|\Psi_n\rangle = e_n|\Psi_n\rangle \qquad e_n = \tfrac{1}{2}an(n-1) + bn. \tag{117}$$

Particular realizations of (117) in terms of one-dimensional solvable potentials were previously considered in [9, 10, 12, 37, 38]. Following the developments in [12], we consider the following remarkable cases.

(i) For $(a = 0, b > 0)$, the spectrum of H_0 is infinite dimensional ($n \in \mathbb{N}$) and does not present degeneracies.

(ii) For $(a > 0, b \geqslant 0)$, the spectrum of H_0 is infinite dimensional ($n \in \mathbb{N}$) and does not present degeneracies.

(iii) For $(a < 0, b \geqslant 0)$, the spectrum of H_0 is finite dimensional with $n = 0, 1, \ldots, s - 1$, where

$$s = -\frac{b}{a} + \frac{3}{2} \qquad \text{for} \quad -2\frac{b}{a} \text{ odd} \tag{118}$$

$$s = -\frac{b}{a} + 1 \qquad \text{for} \quad -2\frac{b}{a} \text{ even,} \tag{119}$$

and all the states are nondegenerate.

It is possible to find a realization of each of the three cases above in terms of exactly solvable dynamical systems in one dimension. We give below the corresponding potential v_0 and transfer operators.

(i) The case $(a = 0, b = 1)$ corresponds to the harmonic oscillator (for which $n \in \mathbb{N}$) with

$$v_0(x) = \tfrac{1}{2}(x^2 - 1) \tag{120}$$

and

$$x^\pm \equiv a^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + x\right). \tag{121}$$

(For the harmonic oscillator, U reduces to the identity operator.)

(ii) The case $(a = 1, 2b = u + v + 1)$, with $u > 1$ and $v > 1$, corresponds to the Pöschl–Teller potential (for which $n \in \mathbb{N}$) with

$$v_0(x) = \frac{1}{8}\left[\frac{u(u-1)}{\sin^2 \frac{x}{2}} + \frac{v(v-1)}{\cos^2 \frac{x}{2}}\right] - \frac{1}{8}(u+v)^2 \tag{122}$$

and

$$x^\pm = \frac{1}{\sqrt{2}}\left[\mp\frac{d}{dx} + \frac{1}{2}\left(u \cot \frac{x}{2} - v \tan \frac{x}{2}\right)\right]. \tag{123}$$

(iii) The case $(a = -1, 2b = 2l - 1)$, with $l \in \mathbb{N}^*$, corresponds to the Morse potential (for which $n = 0, 1, \ldots, l$) with

$$v_0(x) = \tfrac{1}{2}[e^{-2x} - (2l+1)e^{-x} + l^2], \tag{124}$$

and

$$x^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + l - e^{-x}\right). \tag{125}$$

6.3. Phase states and MUB for exactly solvable systems

From equation (53), we can obtain the phase states for a general quantum system described by a truncated generalized oscillator algebra $\mathcal{A}_{\kappa,s}$. We get

$$|m, \varphi\rangle = \frac{1}{\sqrt{s}} \sum_{n=0}^{s-1} e^{-ie_n\varphi} (q_s)^{nm} |\Psi_n\rangle, \qquad (126)$$

with s sufficiently large for the harmonic oscillator and the Pöschl–Teller systems and $s = l + 1$ for the Morse system. Furthermore, equation (87) provides with a mean to generate MUBs associated with the cases (i)–(iii) of subsection 6.2.

On the other hand, the discrete phase state (60) reads here

$$|\mu, \varphi\rangle = C_0 \sum_{n=0}^{s-1} \frac{1}{\sqrt{E(n)}} e^{-ie_n\varphi} (q_s)^{n\mu} |\Psi_n\rangle, \qquad (127)$$

where the factor $E(n)$ can be calculated in the different cases (i)–(iii). A simple calculation gives the following results in term of the Γ function.

 (i) For the harmonic oscillator potential:

$$E(n) = \Gamma(n + 1). \qquad (128)$$

 (ii) For the Pöschl–Teller potential:

$$E(n) = \frac{\Gamma(n + 1)\Gamma(n + u + v + 1)}{2^n \Gamma(u + v + 1)}. \qquad (129)$$

(iii) For the Morse potential:

$$E(n) = \frac{\Gamma(n + 1)\Gamma(2l)}{2^n \Gamma(2l - n)}. \qquad (130)$$

It should be mentioned that the discrete phase states given by (127) differ from the coherent states for exactly solvable potentials derived in [10, 34, 36, 38, 39] from supersymmetric quantum mechanics techniques. The notable difference comes from the fact that the states (127) are temporally stable and are labeled by an integer instead of a continuous complex variable as in the coherent states derived in [10, 34, 36, 38, 39]. The states (127) are eigenstates of the operator (55), whereas the coherent states in [10, 34, 36, 38, 39] are obtained from the three standard definitions (involving annihilation operator, displacement operator and uncertainty relation).

7. Concluding remarks

The starting point of this paper is based on the definition of a generalized oscillator algebra $\mathcal{A}_\kappa$. This algebra is interesting in two respects. First, it describes in an unified way some exactly solvable one-dimensional systems having a nonlinear spectrum (for $\kappa \neq 0$) or a linear spectrum (for $\kappa = 0$). As typical examples, these quantum systems correspond to the Pöschl–Teller potential (for $\kappa > 0$), the Morse potential (for $\kappa < 0$) and the infinite square well potential (for $\kappa = 1/3$) in addition to the harmonic oscillator potential (for $\kappa = 0$). Second, the algebra $\mathcal{A}_\kappa$ can take into account some nonlinear effects that may occur in the quantum description of quantized modes of the electromagnetic field (cf [40]).

In connection with the algebra $\mathcal{A}_\kappa$, the present work addresses three problems: the construction of a phase operator, the determination of its temporally stable eigenstates (the so-called phase states) and the derivation of MUBs from the obtained phase states. This is

the first time that a connection between MUBs and dynamical systems is established. In this regard, the character 'temporally stable' of the eigenstates of the phase operator is essential for the derivation of MUBs. The main results of this paper are as follows.

For the case $\kappa \geqslant 0$ (which corresponds to an infinite representation of $\mathcal{A}_\kappa$), the phase operator is not unitary. We note in passing that the corresponding phase states are similar to those derived in [17] except that our states are temporally stable. However, for $\kappa \geqslant 0$, by making a (*à la* Pegg and Barnett) truncation, which gives rise to a truncated generalized oscillator algebra $\mathcal{A}_{\kappa,s}$, we can define a unitary phase operator whose eigenstates lead to MUBs.

For the case $\kappa < 0$ (which corresponds to a finite representation of $\mathcal{A}_\kappa$), it is possible to construct a unitary phase operator whose eigenstates are temporally stable. MUBs can be derived as a subset of these states. For $\kappa < 0$, the consideration of a truncated generalized oscillator algebra $\mathcal{A}_{\kappa,s}$ is nevertheless necessary in order to establish a connection with the Morse system and to derive associated MUBs.

As a conclusion, in both cases ($\kappa \geqslant 0$ and $\kappa < 0$), the truncation procedure makes it possible to define a unitary phase operator for exactly solvable systems and to generate temporally stable phase states from which MUBs can be derived.

Another result of this paper concerns a new type of phase states. These temporally stable phase states, namely, the states (60), are associated with the truncated algebra $\mathcal{A}_{\kappa,s}$. They are eigenstates of an operator defined in the enveloping algebra of $\mathcal{A}_{\kappa,s}$ and constitute discrete analogs of the coherent states derived in [8]. More generally, this result shows that it is possible, for a finite spectrum, to derive new phase states similar to the coherent states of [8] constructed, for an infinite spectrum, as eigenstates of an annihilation operator. The key of the derivation of the new states (for a finite spectrum) is to add a power of the creation operator to the annihilation operator.

To close this paper, let us mention that the concept of MUBs was recently extended to infinite-dimensional Hilbert spaces [41]. In this vein, it is hoped that the temporally stable phase states derived in this work for the infinite-dimensional case could serve as a hint for deriving MUBs for continuous variables, a difficult challenge.

Acknowledgments

References

[1] Louisell W H 1963 *Phys. Lett.* **7** 60
[2] Susskind L and Glogower J 1964 *Physics* **1** 49
[3] Carruthers P and Nieto M M 1968 *Rev. Mod. Phys.* **40** 411
[4] Pegg D T and Barnett S M 1989 *Phys. Rev.* A **39** 1665
[5] Vourdas A 1990 *Phys. Rev.* A **41** 1653
[6] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[7] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[8] Gazeau J-P and Klauder J R 1999 *J. Phys. A: Math. Gen.* **32** 123
[9] Antoine J-P, Gazeau J-P, Monceau P, Klauder J R and Penson K A 2001 *J. Math. Phys.* **42** 2349
[10] El Kinani A H and Daoud M 2001 *J. Phys. A: Math. Gen.* **34** 5373
     Daoud M and Hussin V 2002 *J. Phys. A: Math. Gen.* **35** 7381
     El Kinani A H and Daoud M 2002 *J. Math. Phys.* **43** 714

[11] Daoud M and Kibler M R 2002 A fractional supersymmetric oscillator and its coherent states *Proc. 6th Int. Wigner Symposium 1999* (Istanbul: Bogazici University Press)

　　　Daoud M and Kibler M R 2001 On fractional supersymmetric quantum mechanics: the fractional supersymmetric oscillator *Symmetry and Structural Properties of Condensed Matter* ed T Lulek, B Lulek and A Wal (Singapore: World Scientific)

[12] Daoud M and Kibler M 2004 *Phys. Lett.* A **321** 147

　　　Daoud M and Kibler M R 2006 *J. Math. Phys.* A **47** 122108

[13] Kibler M R 2006 *Int. J. Mod. Phys.* B **20** 1792

　　　Kibler M R and Planat M 2006 *Int. J. Mod. Phys.* B **20** 1802

　　　Albouy O and Kibler M R 2007 *SIGMA* **3** 076

[14] Kibler M R 2008 *J. Phys. A: Math. Theor.* **41** 375302

　　　Kibler M R 2009 *J. Phys. A: Math. Theor.* **42** 353001

[15] Quesne C and Vansteenkiste N 1998 *Phys. Lett.* A **240** 21

　　　Quesne C 2000 *Phys. Lett.* A **272** 313

　　　Quesne C 2000 *Phys. Lett.* A **275** 313

　　　Quesne C and Vansteenkiste N 2000 *Int. J. Theor. Phys.* **39** 1175

　　　Quesne C 2003 *Mod. Phys. Lett.* A **18** 515

[16] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics, Analysis of Operators* vol 4 (New York: Academic)

[17] Vourdas A, Brif C and Mann A 1996 *J. Phys. A: Math. Gen.* **29** 5887

[18] Vourdas A 1993 *Phys. Scr.* **48** 84

[19] Daoud M, Hassouni Y and Kibler M 1998 The k-fermions as objects interpolating between fermions and bosons *Symmetries in Science X* ed B Gruber and M Ramek (New York: Plenum)

　　　Daoud M, Hassouni Y and Kibler M 1998 *Phys. At. Nuclei* **61** 1821

[20] Ivanović I D 1981 *J. Phys. A: Math. Gen.* **14** 3241

[21] Wootters W K and Fields B D 1989 *Ann. Phys., NY* **191** 363

[22] Berndt B C and Evans R J 1981 *Bull. Am. Math. Soc.* **5** 107

　　　Berndt B C, Evans R J and Williams K S 1998 *Gauss and Jacobi Sums* (New York: Wiley)

[23] Hannay J H and Berry M V 1980 *Physica* D **1** 267

　　　Matsutani S and Ônishi Y 2003 *Found. Phys. Lett.* **16** 325

　　　Rosu H C, Treviño J P, Cabrera H and Murguía J S 2006 *Int. J. Mod. Phys.* B **20** 1860

　　　Merkel W *et al* 2006 *Int. J. Mod. Phys.* B **20** 1893

[24] Witten E 1981 *Nucl. Phys.* B **188** 513

[25] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267

　　　Cooper F, Khare A and Sukhatme U 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific)

[26] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)

[27] Combescure M, Gieres F and Kibler M 2004 *J. Phys. A: Math. Gen.* **37** 10385

[28] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21

[29] Mielnik B 1984 *J. Math. Phys.* **25** 3387

　　　Negro J, Nieto L M and Rosas-Ortiz O 2000 *J. Phys. A: Math. Gen.* **33** 7207

　　　Mielnik B and Rosas-Ortiz O 2004 *J. Phys. A: Math. Gen.* **37** 10007

[30] Fernandez C D J 2009 arXiv:0910.0192

[31] Faddeev L D 1963 *J. Math. Phys.* **4** 72

[32] Pursey D L 1986 *Phys. Rev.* D **33** 2267

[33] Samsonov B F 2000 *J. Phys. A: Math. Gen.* **33** 591

[34] Kumar M S and Khare A 1996 *Phys. Lett.* A **217** 73

[35] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374

[36] Fernandez C D J, Nieto L M and Rosas-Ortiz O 1995 *J. Phys. A: Math. Gen.* **28** 2693

　　　Fernandez C D J and Hussin V 1999 *J. Phys. A: Math. Gen.* **32** 3603

　　　Carballo J M, Fernandez C D J, Negro J and Nieto L M 2004 *J. Phys. A: Math. Gen.* **37** 10349

[37] Quesne C 1999 *J. Phys. A: Math. Gen.* **32** 6705

[38] Angelova M and Hussin V 2008 *J. Phys. A: Math. Theor.* **41** 304016

[39] Bagrov V G and Samsonov B F 1996 *J. Phys. A: Math. Gen.* **29** 1011

[40] Walls D F 1983 *Nature* **306** 141

[41] Weigert S and Wilkinson M 2008 *Phys. Rev.* A **78** 020303